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# Notes on discrete subgroups of $\mathrm{PU}(1,2;\mathbb{C})$ with Heisenberg translations IV (Hyperbolic Spaces and Discrete Groups II)

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# Notes on discrete subgroups of $PU(1, 2; \mathbb{C})$ with Heisenberg translations IV

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## 1. Introduction

In the study of discrete groups it is important to find out conditions for a group to be discrete. We concern ourselves with subgroups of  $PU(1, 2; \mathbb{C})$ . By using the stable basin theorem, Basmajian and Miner have shown

Theorem 1.1 ([1; Theorem 9.11]). *Fix a stable basin point  $(r, \varepsilon)$ . Let  $g$  be a Heisenberg translation of  $PU(1, 2; \mathbb{C})$  with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a} \\ a & 0 & 1 \end{pmatrix},$$

where  $\operatorname{Re}(s) = \frac{1}{2}|a|^2$ . If  $f$  is a loxodromic element of  $PU(1, 2; \mathbb{C})$  with fixed points 0 and  $q$ , satisfying  $|\lambda(f) - 1| < \varepsilon$  and

$$(*) \quad \delta(0, q) > \frac{\delta(0, g(0))}{r^2} (1 + r^2 + \sqrt{1 + r^2}),$$

then the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.

Parker has independently proved the following theorem in a different manner from Basmajian and Miner's.

Theorem 1.2 ([10; Theorem 2.1]). *Let  $g$  be the same Heisenberg translation as in Theorem 1.1. Let  $f$  be any element of  $PU(1, 2; \mathbb{C})$  with isometric sphere of radius  $R_f$ . If*

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty))\delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.

At first sight it is not clear what the relation between these results is. In our previous papers [8] and [9] we have proved that Theorem 1.1 follows from Theorem 1.2. The assumption (\*) in Theorem 1.1 is rather strong and we would like to be able to replace it with a weaker and more geometrical condition. So far we have not been able to do this for all stable basin points. However, by placing additional restriction on  $(r, \varepsilon)$  we show that (\*) may be replaced with a weaker condition. The assumption (\*) in Theorem 1.1 is

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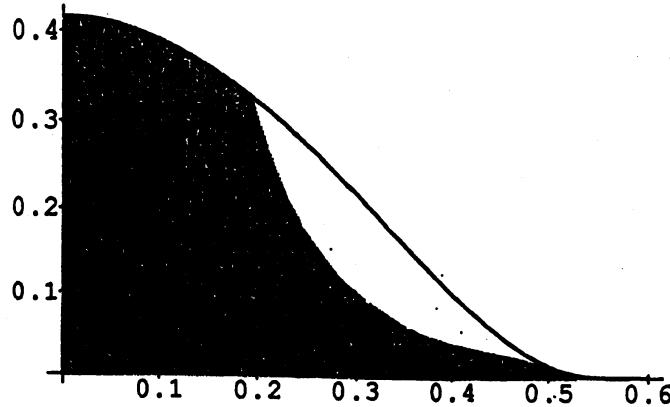
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closely related to a condition on the cross ratio as shown in section 4. Let  $D$  be the set of stable basin points  $(r, \varepsilon)$  such that

$$\frac{1-r}{r} > (2\varepsilon)^{\frac{1}{2}} \left\{ 2 + \left( 8 + \frac{M(\varepsilon)}{2} \right)^{\frac{1}{2}} \right\},$$

where  $M(\varepsilon) = (1+\varepsilon)^{\frac{1}{2}} + (1+\varepsilon)^{-\frac{1}{2}}$ .

The shading in the following figure indicates the set  $D$ .



We have

**Theorem 1.3.** *Fix a stable basin point  $(r, \varepsilon)$  in  $D$ . Let  $g$  be the Heisenberg translation as in Theorem 1.1. If  $f$  is a loxodromic element of  $PU(1, 2; \mathbb{C})$  with fixed points 0 and  $q$ , satisfying  $|\lambda(f) - 1| < \varepsilon$  and  $\| [0, q, g(0), g(q)] \| < r^4$ , then the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.*

## 2. Preliminaries

We recall some definitions and notation. Let  $\mathbb{C}$  be the field of complex numbers. Let  $V = V^{1,2}(\mathbb{C})$  denote the vector space  $\mathbb{C}^3$ , together with the unitary structure defined by the Hermitian form

$$\tilde{\Phi}(z^*, w^*) = -(\overline{z_0^*} w_1^* + \overline{z_1^*} w_0^*) + \overline{z_2^*} w_2^*$$

for  $z^* = (z_0^*, z_1^*, z_2^*)$ ,  $w^* = (w_0^*, w_1^*, w_2^*)$  in  $V$ . An automorphism  $g$  of  $V$ , that is a linear bijection such that  $\tilde{\Phi}(g(z^*), g(w^*)) = \tilde{\Phi}(z^*, w^*)$  for  $z^*, w^*$  in  $V$ , will be called a unitary transformation. We denote the group of all unitary transformations by  $U(1, 2; \mathbb{C})$ . Let  $V_0 = \{w^* \in V \mid \tilde{\Phi}(w^*, w^*) = 0\}$  and  $V_- = \{w^* \in V \mid \tilde{\Phi}(w^*, w^*) < 0\}$ . It is clear that both  $V_0$  and  $V_-$  are invariant under  $U(1, 2; \mathbb{C})$ . We denote  $U(1, 2; \mathbb{C})/(\text{center})$  by  $PU(1, 2; \mathbb{C})$ . Set  $V^* = V_- \cup V_0 - \{0\}$ . Let  $\pi : V^* \rightarrow \pi(V^*)$  be the projection map defined by  $\pi(w_0^*, w_1^*, w_2^*) = (w_1, w_2)$ , where  $w_1 = w_1^*/w_0^*$  and  $w_2 = w_2^*/w_0^*$ . We write  $\infty$  for  $\pi(0, 1, 0)$ . We may identify  $\pi(V_-)$  with the Siegel domain

$$H^2 = \{w = (w_1, w_2) \in \mathbb{C}^2 \mid \operatorname{Re}(w_1) > \frac{1}{2}|w_2|^2\}.$$

We can regard an element of  $PU(1, 2; \mathbb{C})$  as a transformation acting on  $H^2$  and its boundary  $\partial H^2$  (see [6]). Denote  $H^2 \cup \partial H^2$  by  $\overline{H^2}$ . We define a new coordinate system in  $\overline{H^2} - \{\infty\}$ . Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The  $H$ -coordinates of a point  $(w_1, w_2) \in \overline{H^2} - \{\infty\}$  are defined by  $(k, t, w_2)_H \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C}$  such that  $k = \operatorname{Re}(w_1) - \frac{1}{2}|w_2|^2$  and  $t = \operatorname{Im}(w_1)$ . For simplicity, we write  $(t_1, w')_H$  for  $(0, t_1, w')_H$ .

The Cygan metric  $\rho(p, q)$  for  $p = (k_1, t_1, w')_H$  and  $q = (k_2, t_2, W')_H$  is given by

$$\rho(p, q) = \left| \left\{ \frac{1}{2}|W' - w'|^2 + |k_2 - k_1| \right\} + i\{t_1 - t_2 + \operatorname{Im}(\overline{w'}W')\} \right|^{\frac{1}{2}}.$$

We note that the Cygan metric  $\rho$  is a generalization of the Heisenberg metric  $\delta$  in  $\partial H^2$  and that  $\rho$  is invariant under Heisenberg translations (see [7]).

Let  $f = (a_{ij})_{1 \leq i, j \leq 3}$  be an element of  $PU(1, 2; \mathbb{C})$  with  $f(\infty) \neq \infty$ . We define the isometric sphere  $I_f$  of  $f$  by

$$I_f = \{w = (w_1, w_2) \in \overline{H^2} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where  $Q = (0, 1, 0)$ ,  $W = (1, w_1, w_2)$  in  $V^*$  (see [4]). It follows that the isometric sphere  $I_f$  is the sphere in the Cygan metric with center  $f^{-1}(\infty)$  and radius  $R_f = \sqrt{1/|a_{12}|}$ , that is,

$$I_f = \left\{ z = (k, t, w')_H \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Given four distinct points  $q_1, q_2, q_3, q_4$  of  $\partial H^2$ , we define the cross ratio of these points as

$$|[q_1, q_2, q_3, q_4]| = \frac{\delta(q_3, q_1)^2 \delta(q_4, q_2)^2}{\delta(q_4, q_1)^2 \delta(q_3, q_2)^2}.$$

We note that the cross ratio is invariant under  $PU(1, 2; \mathbb{C})$ . The definition is extended by continuity to the case when one of the  $q_i$  is  $\infty$  so, for example,

$$|[q_1, q_2, \infty, q_4]| = \frac{\delta(q_4, q_2)^2}{\delta(q_4, q_1)^2}.$$

Using the cross ratio, one can formulate in an invariant way what it means for pairs of fixed points to be close.

**Proposition 2.1** ([1; Proposition 7.1]). *Let  $f$  and  $g$  be loxodromic elements with fixed points  $\{q_1, q_2\}, \{q_3, q_4\}$ , respectively. If the cross ratio  $|[q_1, q_2, q_3, q_4]| = r^4 < 1$ , then there exists an element  $h \in PU(1, 2; \mathbb{C})$  such that*

- (1)  $hfh^{-1}$  has fixed points at 0 and  $\infty$ , and
- (2)  $ghg^{-1}$  has fixed points at Cygan distance  $r$  and  $1/r$  from 0.

### 3. Stable basin region

We recall the stable basin region (see [1], [8] and [9]). Let

$$B_r = \{z \in \partial H^2 \mid \delta(z, 0) < r\},$$

and let  $\overline{B}_s^c = \partial H^2 - \overline{B}_s$ . Given  $r$  and  $s$  with  $r < s$ , the pair of open sets  $(B_r, \overline{B}_s^c)$  is said to be *stable* with respect to a set  $S$  of elements in  $PU(1, 2; \mathbb{C})$  if for any element  $g \in S$ ,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s^c.$$

Let  $S(r, \varepsilon)$  denote the family of loxodromic elements  $f$  with fixed points in  $B_r$  and  $\overline{B}_{1/r}^c$ , and satisfying  $|\lambda(f) - 1| < \varepsilon$ . For positive real numbers  $r$  and  $r'$  with  $r < 1/\sqrt{3}$  and  $r' < 1$ , we define  $\varepsilon(r, r')$  by

$$\varepsilon(r, r') = \sup\{|\lambda(f) - 1|\}, \quad (3.1)$$

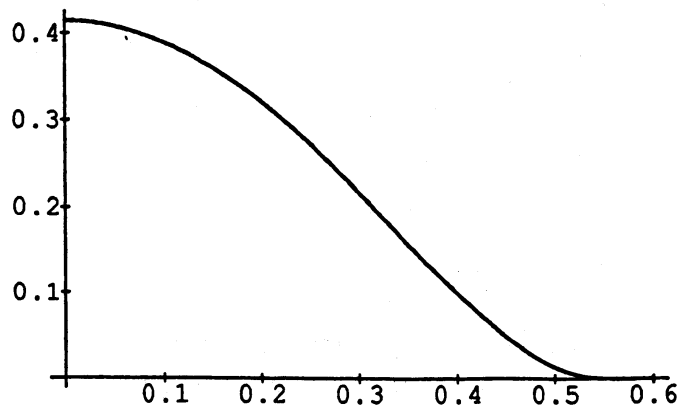
where  $|\lambda(f) - 1|$  satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{1 + \left( \frac{1 - (3 + |\lambda(f) - 1|)r^2}{1 - 2r^2} \right)^2 \left( \frac{1 - 3r^2}{1 - r^2} \right)^2 \left( \frac{r'}{r} \right)^2} - 1. \quad (3.2)$$

A triple of non-negative numbers  $(r, r', \varepsilon)$  is said to be a *basin point* provided that  $r < 1/\sqrt{3}$ ,  $r' < 1$  and  $\varepsilon < \varepsilon(r, r')$ . In particular, if  $r' \leq r$ , we call  $(r, r', \varepsilon)$  a *stable basin point*. Call the set of all such points the *stable basin region*.

**Theorem 3.1** ([9; Theorem 2.2], Stable Basin Theorem). *Given positive real numbers  $r$  and  $r'$  with  $r < 1/\sqrt{3}$  and  $r' < 1$ , the pair of open sets  $(B_{r'}, \overline{B}_{1/r'}^c)$  is stable with respect to the family  $S(r, \varepsilon(r, r'))$ , where  $\varepsilon(r, r')$  is given by (3.1).*

The following figure shows the stable basin region.



#### 4. Groups with Heisenberg translations

In this section we show that Theorem 1.3 follows from Theorem 1.2. To prove Theorem 1.3, we need two lemmas.

**Lemma 4.1.** *Suppose that  $\delta(0, g(0)) < \delta(q, g(q))$ . If  $||[0, q, g(0), g(q)]|| < r^4$ , then*

$$\delta(0, q) > \left( \frac{1-r}{r} \right) \delta(0, g(0)).$$

**Proof.** Using the triangle inequality and the invariance of  $\delta$  under Heisenberg translations, we have

$$\delta(q, g(0)) \leq \delta(0, g(0)) + \delta(0, q)$$

and

$$\delta(0, g(q)) \leq \delta(0, g(0)) + \delta(g(0), g(q)) = \delta(0, g(0)) + \delta(0, q).$$

It follows that

$$\begin{aligned} r^4 &> |[0, q, g(0), g(q)]| \\ &= \left( \frac{\delta(0, g(0))\delta(q, g(q))}{\delta(0, g(q))\delta(q, g(0))} \right)^2 \\ &> \left( \frac{\delta(0, g(0))}{\delta(0, g(0)) + \delta(0, q)} \right)^4, \end{aligned}$$

which implies

$$\delta(0, q) > \left( \frac{1-r}{r} \right) \delta(0, g(0)).$$

**Lemma 4.2** ([9; Lemma 3.3]). *Let  $f$  be a loxodromic element with fixed points 0 and  $q$ , satisfying  $|\lambda(f) - 1| < \varepsilon$ . Then*

$$\left( \frac{\delta(0, q)}{R_f} \right)^2 \leq 2\varepsilon.$$

We are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Without loss of generality, we may assume that  $\delta(0, g(0)) < \delta(q, g(q))$ , because Theorem 1.2 is invariant under Heisenberg translations. Let  $(r, \varepsilon)$  be a stable basin point in  $D$ . By Lemmas 4.1 and 4.2,

$$\begin{aligned}
R_f &> \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \delta(0, q) \\
&> \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \left(\frac{1-r}{r}\right) \delta(0, g(0)) \\
&= \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \left(\frac{1-r}{r}\right) |s|^{\frac{1}{2}} \\
&> \left\{2 + \left(8 + \frac{M(\varepsilon)}{2}\right)^{\frac{1}{2}}\right\} |s|^{\frac{1}{2}} \\
&> \left\{2 + \left(8 + \frac{L}{2}\right)^{\frac{1}{2}}\right\} |s|^{\frac{1}{2}} \\
&= 2|s|^{\frac{1}{2}} + \left(8|s| + \frac{L|s|}{2}\right)^{\frac{1}{2}} \\
&> \sqrt{2}|a| + \left(4|a|^2 + \frac{L|s|}{2}\right)^{\frac{1}{2}}.
\end{aligned}$$

In the same manner as in the proof Theorem 4.5 in [8] we have

$$\begin{aligned}
R_f^2 &> \frac{|s|L}{2} + 2\sqrt{2}|a|R_f + 2|a|^2 \\
&> \delta(gf(\infty), f(\infty))\delta(gf^{-1}(\infty), f^{-1}(\infty)) + 2|a|^2.
\end{aligned}$$

We conclude from Theorem 1.3 that the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.

**Collorary 4.3.** *Fix a stable basin point  $(r, \varepsilon)$  in  $D$ . Let  $g$  be the same Heisenberg translation as in Theorem 1.1. If  $f$  is a loxodromic element with fixed points 0 and  $q$ , satisfying  $|\lambda(f) - 1| < \varepsilon$  and  $\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$ , then the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.*

We need a lemma to prove Collorary 4.3.

**Lemma 4.4** ([1; Lemma 7.3]). *If  $\delta(0, q) > \delta(0, g(0))$ , then*

$$|[0, q, g(0), g(q)]|^{\frac{1}{2}} \leq \left(1 + \frac{\delta(0, q)}{\delta(0, q) - \delta(0, g(0))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right).$$

**Proof of Collorary 4.3.** We see that our assumption

$$\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$$

is equivalent to

$$\left(1 + \frac{\delta(0, q)}{\delta(0, q) - \delta(0, g(0))}\right) \left(\frac{\delta(0, g(0))}{\delta(0, q) - \delta(0, g(0))}\right) < r^2.$$

It follows from Lemma 4.4 that  $||[0, q, g(0), g(q)]|| < r^4$ . By Theorem 1.3, the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is not discrete.

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